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OPTIMAL DESIGN OF PLL WITH TWO SEPARATE PHASE DETECTORS.(U)
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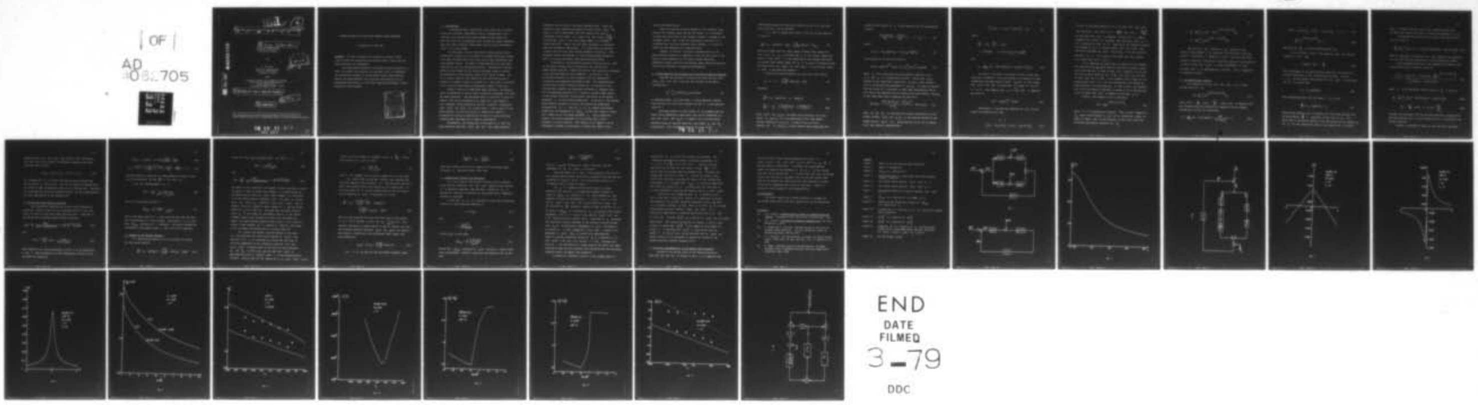

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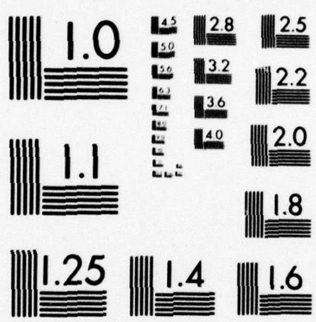

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(6) OPTIMAL DESIGN OF PLL WITH TWO SEPARATE PHASE DETECTORS +

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OPTIMAL DESIGN OF PLL WITH TWO SEPARATE PHASE DETECTORS

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ABSTRACT: In order to improve the operation of phase locked loops in both the acquisition and tracking modes, loops with two phase detectors are considered.

The optimization problem of choosing the phase detectors in order to minimize the variance of loop phase error is formulated. The optimization problem is solved under several approximations. Despite the approximation, the resulting nonlinearities can be implemented and results in a PLL with improved performance in both acquisition and tracking.

1. Introduction.

In designing phase locked loop system (PLL) one is usually faced with different requirements such as minimum phase error variance, small acquisition time, maximum locking range, etc. With the usual way of designing these systems one cannot adequately cope with these different requirements and the total performance often is not good enough.

In this paper, we will show that by adding another phase detector the system can be provided with enough additional freedom to be able to better deal with these conflicting requirements and yield smaller errors. In order to find good forms for the phase detectors we will use optimization theory. The detector function which we obtain might not be strictly optimum, owing to several approximations in the analysis which we are forced to make. Yet, as is clearly shown by our simulation, the resulting system is superior to regular systems in current use. This study is believed to be the first which exploit optimization theory in this way.

Owing to the use of additional phase detector, the problem under consideration is one of (near) optimality designing a PLL with two phase detectors. The equivalent model of such PLL is given in Figure 1 and can be considered as a regular PLL (see Figure 2) with $H(p)g(\phi)$ being replaced by the sum $H_1(p)f(\phi) + H_2(p)f_1(\phi)$, where $H_1(p)$ and $H_2(p)$ are linear filters. Such a split of the phase detector and the loop filter is shown to be useful in designing the system to perform well in both its acquisition and tracking modes and hence has a superior performance.

Consequently, the design goal of the paper is to find the phase detector functions $f_1(\phi)$ and $f(\phi)$ that (approximately) minimizes the variance of the phase tracking error. Since the analysis of this problem with general linear filters $H_1(p)$ and $H_2(p)$, is quite complicated, only the second order case is considered in this work: i.e., $H_1(p) = H_0$ and $H_2(p) = \frac{1-H_0}{1+\tau_1 p}$. Clearly when $f_1(\phi) = f(\phi)$ the system reduces to a standard second order PLL. Hence, with two phase detectors, we obviously have more freedom in the design. It is shown that our solution involves the use of an extra design parameter (termed T_1). This parameter enables the design of a PLL having minimum variance as well as locking time smaller than some prescribed value.

In Section 2, the probability density function (p.d.f) $p(\phi)$ of the phase tracking error is derived as the solution of a Fokker Planck equation. Simplified form of this function is obtained using some approximation and assumption. In Section 3, the minimization problem is stated together with its appropriate constraints and augmented performance index. The (approximately) optimal phase detector functions which are the solution of the related Euler equation are presented in Section 4, together with the corresponding optimal p.d.f. Section 5 includes some remarks on the optimal solution; in particular, the effect of the design parameter T_1 on the system locking time. In Section 6, the optimal phase detector function is firstly calculated and graphically presented for $T_1 = 0$. Then for $T_1 \neq 0$, we compute and present graphically the dependence of $\sigma_\phi^2|_{min}$ on the input signal to noise ratio (SNR) and on the design parameter T_1. Using simulation results, the actual dependence of the locking time on this parameter is also presented. This enables us to suggest a way of designing an optimal system having a locking time smaller than certain prescribed value.

Since the optimal solution depends on the frequency error between the incoming signal and the VCO output, it is plausible to check the sensitivity of the optimal system to a change in this frequency error. This is done by simulations from which it is concluded that, with the frequency error unknown, it is best to design for the maximum of its expected value.

Finally, in Section 7 practical implementation of the phase detector function is suggested, using the first term of the Fourier series expansion of the actual optimal functions. Simulation results with the "sub-optimal" phase detector functions justify the usefulness of proposed approximation.

2. System Equation and Tracking Error Probability Density Function

Consider the problem of selecting two phase detector functions, $f_1(\phi)$ and $f(\phi)$, such that the variance of the phase tracking error

$$\sigma_\phi^2 = \int_{-\pi}^{\pi} (\phi-\overline{\phi})^2 p(\phi, f_1(\phi), f(\phi)) d\phi \tag{1}$$

is minimized where $p(\phi, f_1(\phi), f(\phi))$ is the probability density function (p.d.f) of the error (modulo 2π) and $\overline{\phi}$ is the expected value of ϕ.

Following Lindsey [1] and Viterbi [2] in assuming that the input filter bandwidth is much wider than the PLL bandwidth, we have that $N_f(t)$ and $N_{f_1}(t)$ of Figure 1 are zero mean white Gaussian noises of spectral densities (one-sided) N_0 and N_{0_1} respectively. This assumption enables the usage of the

Fokker-Planck equation from whose solution the p.d.f of the error $p(\phi, f_1(\phi), f(\phi))$, can be obtained.

It is easy to deduce from Figure 1 that the system equation is given by

$$\frac{d\phi}{dt} = \Lambda_0 - H_0(AKf(\phi) + KN_f) - \frac{1-H_0}{1+\tau_1 p}(AKf_1(\phi) + KN_{f_1}) \tag{2}$$

where we assumed that the input signal includes phase modulation $\theta(t) = \Lambda_0 t + \theta_0$, Λ_0 is constant and A is the signal's amplitude. Also, $K = K_v B$ where B is the amplitude of the voltage controlled oscillator (VCO) output signal and K_v is its gain in radian/sec/volt. The loop filters are $H_1(p) = H_0$ and $H_2(p) = (1-H_0)/(1+\tau_1 p)$, where $p \equiv d/dt$ is the Heaviside operator.

To write the system equation in state space form, define

$$y_0 = \phi, \quad y_1 = -\frac{(1-H_0)}{1+\tau_i p}[AKf_1(y_0) + KN_g] \tag{3}$$

and hence

$$\frac{dy_0}{dt} = \Lambda_0 - H_0 AKf(y_0) + y_1 - H_0 KN_g(t) \tag{4a}$$

$$\frac{dy_1}{dt} = -\frac{y_1}{\tau_1} - \frac{(1-H_0)AKf_1(y_0)}{\tau_1} - \frac{(1-H_0)KN_{f_1}(t)}{\tau_1}\,. \tag{4b}$$

Since $N_f(t)$ and $N_{f_1}(t)$ are white noise processes, the state vector $\underline{y} = (y_0, y_1)$ is a two dimensional first order Markov process, whose p.d.f. $p(y_0, y_1)$ a Fokker-Planck satisfies equation [1]. If $p(y_0, y_1)$ is well defined then integrating the equation with respect to y_1 yields equation (5) for the marginal density,

$$\frac{\partial(r(y_0)p(y_0))}{\partial y_0} + \frac{\partial^2 p(y_0)}{\partial y_0^2} = 0 \qquad -\pi \leq y_0 \leq \pi \tag{5}$$

where,

$$r(y_0) = -4(\Lambda_0 - H_0 AKf(y_0) + E(y_1|y_0))/H_0^2 K^2 N_0. \tag{6}$$

The solution of (5) can be written as

$$p(y_0) = c_0 \exp[-\int^{y_0} r(x)dx \ (1+c_1 \int_{-\pi}^{y_0} [\exp \int^{z} r(x)dx]dz) \tag{7}$$

where c_1 can be obtained from the boundary condition $p(\pi) = p(-\pi)$ and c_0 is a normalization factor. Notice that $p(y_0)$ depends on the conditional expectation $E(y_1|y_0)$ which requires for its calculation the knowledge of $p(y_0, y_1)$. In order to proceed from this point it is necessary to make some approximation and we will follow the approach used by Lindsey which approximated $E(y_1|y_0)$ by the right hand side of equation (8) and write

$$E(y_1|y_0) = \frac{AK(1-H_0)(f_1(y_0) - \overline{f}_1)S_{f_1}(0)}{2\tau_1 \sigma_{f_1}^2} - AK(1-H_0)\overline{f}_1 \tag{8}$$

where $\overline{f}_1$ and $\sigma_{f_1}^2$ are the mean and variance respectively of the random variable $f_1(\phi)$, and $S_{f_1}(\omega)$ is the spectral density of the random process $(f_1(\phi) - \overline{f}_1)$. Using equation (8) in (6) we obtain after some algebraic manipulations

$$\int^{\phi} r(x)dx = -\alpha F(\phi) + \beta F_1(\phi)/\sigma_{f_1}^2 - \beta_1\phi \tag{9}$$

where,

$$\frac{dF}{d\phi} = -f(\phi), \quad \frac{dF_1}{d\phi} = -f_1(\phi),$$

$$\alpha = 4A/H_0KN, \qquad \beta = 2A(1-H_0)S_{f_1}(0)/\tau_1N_0H_0^2K$$

and

$$\beta_1 = \frac{\alpha}{AKH_0}[\Lambda_0 - AK(1-H_0)\overline{f}_1(\phi)(1 + \beta H_0/\alpha(1-H_0)\tau_{f_1}^2)]. \tag{10}$$

In order to facilitate a solution we need to make some additional assumption. The restrictions to be imposed by these assumptions will make physical sense. In fact, we will assume that $f_1(\phi)$ and $f(\phi)$ are such that $\int_{-\phi}^{\phi} r(x)dx = 0$ for all $\phi \in [-\pi,\pi]$. This implies that $c_1 = 0$ and $p(\phi)$ is symmetric and given by

$$p(\phi) = c_0\exp[-\int^{\phi} r(x)dx]. \tag{11}$$

Furthermore, to satisfy the condition on $r(x)$, we will assume (see equation (9)) that

$$\beta_1 = 0$$

$$\alpha F(\phi) - \beta F_1(\phi)/\sigma_{f_1}^2 = \alpha F(-\phi) - \beta F_1(-\phi)/\sigma_{f_1}^2. \tag{12}$$

One way of satisfying equation (12) is to make $F(\phi)$ and $F_1(\phi)$ even functions. But, then, $f(\phi) = -\frac{dF(\phi)}{d\phi}$ and $f_1(\phi) = -\frac{dF_1(\phi)}{d\phi}$ are odd and since $p(\phi)$ is symmetric, we have $\overline{f}_1 = 0$. Hence, equation (10) yields $\beta_1 = \alpha\Lambda_0/AKH_0$ and, therefore, with $\beta_1 = 0$ it is necessary to have $\Lambda_0 = 0$. Since we do not intend to restrict our problem to $\Lambda_0 = 0$, we will choose $F(\phi)$ and $F_1(\phi)$ such that their weighted sum $\alpha F(\phi) - \beta F_1(\phi)/\sigma^2_{f_1}$ is an even function and that $\beta_1 = 0$. It is shown in the sequel that this enables us to treat the case of $\Lambda_0 \neq 0$.

It is important to notice that in contrast to the standard PLL case where the one phase detector function is usually chosen as an odd function, in order to make $p(\phi)$ symmetric [1],[3],[4], for the two phase detector case $F'(\phi)$ and $F_1'(\phi)$, are not odd. In fact, when $\Lambda_0 \neq 0$, $p(\phi)$ is not symmetric with respect to the steady-state error $\phi \neq 0$ for the standard PLL even though the phase detector is an odd function. In view of this, the weighted sum of the two detector functions is chosen to be an odd function so as to ensure that $p(\phi)$ is symmetric even when $\Lambda_0 \neq 0$.

Finally, when equation (12) is satisfied,

$$p(\phi) = \frac{1}{D^*} e^{\alpha F(\phi) - F_1(\phi)/\sigma^2_{f_1}} \tag{13}$$

where D^* is a normalization constant. That is, $p(\phi)$ depends on $\sigma^2_{f_1}$, which itself depends on $p(\phi)$ by its definition. Hence, in order to obtain $p(\phi)$ be well defined, we must first solve the following simultaneous equations for $\sigma^2_{f_1}$

$$\sigma_{f_1}^2 = \frac{1}{D^*}\int_{-\pi}^{\pi}(f_1(\phi) - \bar{f}_1)^2 e^{\alpha F(\phi)-\beta F_1(\phi)/\sigma_{f_1}^2} d\phi \tag{14}$$

$$\bar{f}_1 = \frac{1}{D^*}\int_{-\pi}^{\pi} f_1(\phi) e^{\alpha F(\phi)- F_1(\phi)/\sigma_{f_1}^2} d\phi .$$

The fact that $p(\phi)$ depends on $\sigma_{f_1}^2$ complicates the search of the minimum of σ_ϕ^2 which will be discussed in the next section. Furthermore, β depends on $S_{f_1}(0)$ and since the latter cannot be explicitly calculated, a heuristic approach (see Section 5) is used to argue that $S_{f_1}(0)$ is almost independent of $f_1(\phi)$ and hence that β is approximately a constant. This is supported by simulation results.

3. The Minimization Problem

The problem is to find $f(\phi)$ and $f_1(\phi)$ so as to have minimum value for σ_ϕ^2

$$\sigma_\phi^2 = \frac{1}{D^*}\int_{-\pi}^{\pi}\phi^2 e^{\alpha F(\phi)-\beta F_1(\phi)/\sigma_{f_1}^2} d\phi \tag{15}$$

where, $f(\phi) = -\frac{dF}{d\phi}$, $f_1(\phi) = -\frac{dF_1}{d\phi}$. Since $p(\phi)$ in equation (13) was obtained using the conditions of equation (12) the optimal problem must satisfy the constraints

$$\beta_1 = \frac{\alpha}{AKH_0}[\Lambda_0 - AK(1-H_0)\bar{f}_1(1 + \frac{H_0\beta}{\alpha(1-H_0)\sigma_{f_1}^2})] = 0 \tag{16}$$

$$\alpha F(\phi) - \beta F_1(\phi)/\sigma_{f_1}^2 = \alpha F(-\phi) - \beta F_1(-\phi)/\sigma_{f_1}^2 \qquad -\pi \leq \phi \leq \pi \tag{17}$$

$$\frac{1}{D^*}\int_{-\pi}^{\pi} e^{\alpha F(\phi) - F_1(\phi)/\sigma_{f_1}^2} dy = 1. \tag{18}$$

The value of $\sigma_{f_1}^2$ is obtained from equation (14).

It is possible to show, using equations (16), (12) and the symmetry of $p(\phi)$ that

$$(1-H_0)\overline{F}_1' + H_0\overline{F}' = -\frac{\Lambda_0}{AK} \tag{19}$$

This requirement on the phase detector functions will prove to be of great importance in the system acquisition mode. In fact, $t >> \tau_1$ one may deduce from equation (3) that $F_1'(\phi)$ is stationary and $\overline{F}_1'$ is constant and

$$-\overline{y}_1 = (1-H_0)AK\overline{F}_1'(\phi). \tag{20}$$

Now averaging equation (4a) and using $\phi = y_0$, we get

$$\frac{\overline{d\phi}}{dt} = \Lambda_0 + H_0AK\overline{F}'(\phi) + \overline{y}_1. \tag{21}$$

Substituting equation (20) in equation (21) and using equation (19) we end up having $\frac{d\overline{\phi}}{dt} \to 0$ regardless of the value of Λ_0. This means that by satisfying condition (19) the system ends up locking for any Λ_0 and the acquisition range of this general PLL is infinite.

This is certainly not the case for the standard second order PLL where the acquisition range is limited and depends on H_0 [1].

To conclude this discussion, we notice that equation (19) can be written as

$$-\frac{\Lambda_0}{AK} = \frac{1}{D^*}\int_{-\pi}^{\pi}[H_0F'(\phi) + (1-H_0)F_1'(\phi)]\exp(\alpha F(\phi) - \beta F_1(\phi)/\sigma^2_{f_1})d\phi \tag{22}$$

which is equivalent to equation (16). This together with equations (17) and (18) form the constraints of the optimization problem.

Using Lagrange multipliers, the augmented performance index associated with this problem is given by

$$J = \frac{\int_{-\pi}^{\pi}[\phi^2 + \lambda_1(H_0F'(\phi) + (1-H_0)F_1'(\phi) + \frac{\Lambda_0}{AK})]e^{\alpha F(\phi) - F_1(\phi)/\sigma^2_{f_1}}}{\int_{-\pi}^{\pi}\exp[\alpha F(\phi) - \beta F_1(\phi)/\sigma^2_{f_1}]d\phi} \tag{23}$$

where λ_1 is the multiplier and the value of $\sigma^2_{f_1}$ is given by

$$\sigma^2_{f_1} = \frac{1}{D^*}\int_{-\pi}^{\pi}(f_1'(\phi) - \overline{F}_1')^2\exp[\alpha F(\phi) - \beta F_1(\phi)/\sigma^2_{f_1}]d\phi \tag{24a}$$

$$\overline{F}_1' = -\frac{\Lambda_0}{AKa}, \text{ where } a = (1-H_0) + \frac{H_0\beta}{\alpha\sigma^2_{f_1}}. \tag{24b}$$

Although the constraint (17) has not been explicitly considered in deriving equation (23), it can be shown that it is satisfied in the optimal solution.

Because of shortage of space, we will not write the Euler equaton which $F(\phi)$ and $F_1(\phi)$ must satisfy, (for full detail see [3]), and we only mention the boundary condition that these functions must satisfy:

$$F_1(\pi) - F_1(-\pi) \triangleq T_1, \quad F'(\pi) = F'(-\pi). \tag{25}$$

For standard PLL, $T_1 = 0$, while with the two phase detector PLL, we can choose T_1 arbitrarily. This adds a degree of freedom which will prove to have an important effect on the locking time. Therefore, we can design the system for minimum variance σ_ϕ^2 and simultaneously put some requirements on the locking time.

4. The Optimal Phase Detector Function

The system Euler equations are second order differential equations. However, since we are interested only in $F'(\phi)$ and $F_1'(\phi)$, we need to solve first order equations only. From this we can obtain the two optimal phase detector functions

$$F_1'(\phi) - \overline{F}_1' = \frac{H_0\beta/\alpha}{(T_1/2\pi + \Lambda_0/AKa)(a-1+H)}\left[-1 + \frac{|x|}{\pi}\sqrt{\phi^2 + \pi^2/\sinh^2 x}\right] \tag{26}$$

and

$$F'(\phi) = \frac{a-1+H_0}{H_0} F_1'(\phi) - \frac{\phi/\alpha}{\phi^2+\pi^2/\sinh^2 x} \tag{27}$$

which depend on the system parameters as well as on the parameters a and x. These parameters are the simultaneous solution of the two nonlinear equations,

$$\frac{a}{2}\,(T_1/2 + \Lambda_0/AKa) = (a-1+H_0)\left[\frac{T_1/2\pi}{x\coth x} + \frac{\Lambda_0}{AKa}\right] \tag{28}$$

$$-2\left(1 + \frac{(T_1/2\pi + \Lambda_0/AKa)^2(a-1+H_0)\alpha}{\beta H_0}\right)\frac{\sinh^2}{x^2} + \frac{\sinh 2x}{2x} + 1 = 0. \tag{29}$$

The phase detector functions are (approximately) optimal in the sense of minimizing σ_ϕ^2 and $\frac{d\bar{\phi}}{dt} \to 0$ for every Λ_0.

Also, the corresponding p.d.f. is

$$p(\phi) = \frac{1}{2|x|}\;\frac{1}{\sqrt{\phi^2 + \pi^2/\sinh^2 x}}\,, \tag{30}$$

and the corresponding variance is

$$\sigma_\phi^2\big|_{min} = \frac{\pi^2}{2}\left[\frac{\coth x}{x} - \frac{1}{\sinh^2 x}\right] \tag{31}$$

Notice that many values of x solve equation (28) and (29) each one corresponding to a local minimum. From Figure 3, it is clear that $\sigma_\phi^2|_{min}$ decreases as x increases. The global minimum will correspond to the highest value x that solve these equations.

5. Remarks on the Optimal Solution

(1) Substituting equation (27) in equation (2) we have for the system equation

$$\frac{d\phi}{dt} = \Lambda_0 - H_0 AKf_D(\phi) - \frac{1+\tau_2^* p}{1+\tau_1 p}\; a\, AKf_1(\phi) - H_0 KN^* \tag{32}$$

where $N^* = N_f + N_{f_1}(1-H_0)/H_0(1+\tau_1 p)$; $\tau_2^* = \frac{a-1}{a}\tau_1 + \tau_2$

$$f_D(\phi) = \frac{\phi/\alpha}{\phi^2+\pi^2/\sinh^2 x} \tag{33}$$

and

$$f_1(\phi) = \frac{\Lambda_0}{AKa} - \frac{H_0\beta/\alpha}{(T_1+2+\Lambda_0/AKa)(a-1+H_0)}\left[-1 + \frac{|x|}{\pi}\sqrt{\phi^2+\pi^2/\sinh^2 x}\right]. \tag{34}$$

The model of the system that corresponds to these equations is given in Figure 4, where we notice that channel 1 is controlled by an odd phase detector function $f_D(\phi)$, while channel 2 is controlled by an even phase detector function $f_1(\phi)$. For small ϕ, we have $f_D(\phi) \simeq a_d$ where $a_d = \sinh^2 x/\pi^2\alpha$ and $f_1(\phi) \simeq$ const. That is channel 2 does not respond to small changes in ϕ and the system behaves as a first order PLL. Noticing that the phase detector slope a_d is very large for reasonable value of x, our results resemble similar ones obtained by Stiffler [4] and Shaft [5], namely, the phase detector function that causes a first order PLL system to have minimum σ_ϕ^2 is sgn(sin σ). However, with large or PLL unlocked, the system acts as a second order PLL.

(2) Clearly the optimal solution depends on the ratio α/β (see equation (29) which itself depends on $S_{f_1}(0)$ (see equation (9)). It is reasonable to assume that the ratio of the noise bandwidths of the stochastic processes $\phi(t)$ and $f_1(\phi) - \bar{f}_1(\phi)$ is proportional to the ratio of their variances σ_ϕ^2 and $\sigma_{f_1}^2$. In this case, one can show that $S_f(0)$ is approximately given by $\eta N_0/2A^2$, where η is the proportionality constant. (Notice that for the regular PLL (i.e. $f_1(\phi) = f(\phi) = \sin\phi$)

Viterbi [2] and Lindsey [1] estimated $S_f(0)$ by $\frac{N_0}{2A_2}$.) Using this estimate for $S_f(0)$ we have

$$\beta/\alpha = \frac{1-H_0}{H_0} \frac{N_0}{4\tau_1 A^2} . \tag{35}$$

That is β/α depends on the noise to signal ratio of the input.

(3) Notice that the optimal solution (see equations (28) and (29)) depends on the parameter T_1. Recalling that this parameter (see equation (25)) must vanish for the standard PLL. To show the effect of T_1 on the locking time; we use the Fourier series expansion of $f_1(\phi)$ in equation (32) to get

$$\frac{d\phi}{dt} = [\Lambda_0 + \frac{(1+\tau_2^* p)a}{1+\tau_1 p} \frac{AKT_1}{2\pi}] - H_0 AK f_D(\phi)$$
$$- \frac{(1+\tau_2^* p)a}{1+\tau_1 p} AK\, g_D(\phi) - H_0 KN^* \tag{36}$$

where we also used the fact that the first term of this series is $a_0 = 1/2\pi \int f_1(\phi)d\phi = -T_1/2\pi$, and $g_D = \sum_{n=1}^{\infty} a_n \cos n\phi$. This equation represents a system similar to that of Figure 4 but with the two phase detector functions $f_D(\phi)$ and $g_D(\phi)$ for channel 1 and channel 2 respectively, and equivalent input signal with phase modulation

$$\theta_{eq}(p) = \frac{1}{p} [\Lambda_0 + \frac{(1+\tau_2^* p)}{1+\tau_1 p} AKaT_1/2\pi] . \tag{37}$$

For $t \gg T_1$, we have for the equivalent frequency input

$$\frac{d\theta_{eq}(t)}{dt} = \Lambda_0(1 + \frac{aT_1/2\pi}{\Lambda_0/AK}). \tag{38}$$

Since the locking time of a PLL depends on the incoming signal frequency, T_1 obviously affects this time.

6. Computational Results and Simulation

As it was shown in the previous section, the two optimal phase detector functions $f(\phi)$ and $f_1(\phi)$ depend on the solution of the nonlinear equations (28) and (29). Except for $T_1 = 0$, an analytic solution is difficult to obtain and hence a numerical solution is obtained.

In fact for $T_1 = 0$, it is possible to show that the desired solution to these nonlinear equations is

$$a = 2(1-H_0) \tag{39}$$

and

$$x \simeq \frac{(\Lambda_0 AK)^2 \alpha}{2(1-H_0)H_0\beta} \qquad x >> 1. \tag{40}$$

In this case, we also have

$$\sigma_\phi^2|_{min} = \frac{\pi^2}{2}\frac{(1-H_0)H_0\beta/\alpha}{(\Lambda_0/AK)^2} \tag{41}$$

Notice that $\sigma_\phi^2|_{min}$ decreases as Λ_0/AK increases, a result which seems unreasonable. However, using the loop equation (32), we can have

$$\frac{\Lambda_{eff}}{AK} = - \frac{4(1-H_0)H_0\beta/\alpha}{\Lambda_0/AK} . \tag{42}$$

That is, Λ_{eff}/AK decreases as Λ_0/AK increases, and the dependence of σ_ϕ^2 on Λ_0/AK is justified.

With the values of a and x from equations (40) and (41) the optimal phase detector functions of equations (33) and (34) are presented in Figures 5 and 6. The corresponding optimal p.d.f of equation (30) is given in Figure 7.

For $T_1 \neq 0$ computer search can be used in solving equations (28) and (29). As a numerical example consider in Figure 8, $\sigma_\phi^2|_{min}$ as a function of α/β, (which depends on S/N, equation 35) with system parameters $r = 2$, $H_0 = 0.02$, $T_1 = 0.03$ and for $\Lambda_0/AK = 0.12$ and 0.08. Notice that for a fixed α/β, σ_ϕ^2 decreases as $\frac{\Lambda_0}{AK}$ increases, similar to what we had for $T_1 = 0$. In Figure 9, $\sigma_\phi^2|_{min}$ as a function of T_1 is presented with system parameters $\alpha/\beta = 3$, $r = 2$, $H_0 = 0.01$ and for $\Lambda_0/AK = 0.12$ and 0.08. Notice that for a fixed $\alpha/\beta, \sigma^2$ decreases as T_1 increases. The theoretical results obtained in the previous sections assumed that $S_{f_1}(0)$ is practically independent of $f_1(\phi)$ (see Remark 2 of Section 5). If this assumption is true, then η should be independent of $f_1(\phi)$. Thus, to justify this approximation, Figure 9 includes simulation results of σ_ϕ^2 as a function of T_1, with $\Lambda_0/AK = 0.08$ and 0.12, and for $\eta = 0.08$. Although this value of η was obtained for a single point on the curve, the small difference along the curve between this simulation and the previously computed results do support the assumption.

In Figure 10, simulation results of the locking time as a function of T_1 in a noise free system are presented. The simulation performed for optimal system whose parameters are $r = 2$, $H_0 = 0.01$ $\frac{\Lambda_0}{AK} = 0.12$ and $\alpha/\beta = 3$. Locking time was defined to be the time for the system to reach $|\frac{d\phi}{dt}| < 10^{-4}$. Notice that the locking time has minimum value. Therefore the design goal could be to select T_1 to have the largest value which still satisfies, according to Figure 10, the locking time requirement (i.e., locking time $\leq t_{max}$). It should be emphasized that with the regular PLL, T_1 is necessarily zero and there is no extra freedom in selecting the locking time for a given Λ_0/AK.

Previously, the assumption that Λ_0 is known was used. This is not true, in practice. Hence, it is important to discuss the behaviour of a system, optimally designed to track Λ_0^*, when it is actually tracking $\Lambda_0 \neq \Lambda_0^*$. That is, to find σ_ϕ^2 of a system having phase detector functions $f_1^*(\phi)$ and $f^*(\phi)$. It is practically impossible to establish analytically the p.d.f. $p(\phi)$ of this case. Instead, $\sigma_\phi^2/\sigma_\phi^{2*}$ as a function of Λ_0/Λ_0^* was calculated numerically. The results are depicted in Figures 11 and 12, from which it is noted that $\sigma_\phi^2/\sigma_\phi^{2*}$ is not symmetric with respect to $\Lambda_0/\Lambda_0^* = 1$. It increases for $\Lambda_0/\Lambda_0^* < 1$ much faster than for $\Lambda_0/\Lambda_0^* < 1$. In view of this, we may conclude that for Λ_0/AK not known a priori, it is reasonable to design the system for the maximal expected incoming frequency error.

7. Practical Implementation of the Optimal Phase Detector

Instead of the optimal forms of the functions given on equations (33) and (34) or Figures 5 and 6, it is suggested that the first term of their Fourier expansions be used, i.e., $f_D(\phi) \cong b_1 \sin\phi$ and $f_1(\phi) \cong \frac{a_0}{2} + a_1 \cos\phi$, where b_1, a_0 and a_1 are the Fourier coefficients. To examine this approximation, Figure 13 compares the dependence of σ_ϕ^2 on T_1 of this system with that of the optimal one. It is noted that with this sub-optimal system, the variance of the phase error is not much larger than that of the optimal one, which justifies the usefulness of using these easy to implement phase detector functions. Figure 14 presents a block diagram of such implementation.

Acknowledgment

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Legends

Figure 1. Model of the PLL with two phase detectors

Figure 2. Model of standard PLL

Figure 3. $\sigma_\phi^2|_{min}$ as a function of x

Figure 4. Simplified model of second order PLL with two phase detector functions

Figure 5. The optimal phase detector $f_1(\phi)$ with $T_1 = 0$

Figure 6. The optimal phase detector $f_D(\phi)$ with $T_1 = 0$

Figure 7. The optimal probability density function $p(\phi)$ with $T_1 = 0$

Figure 8. $\sigma_\phi^2|_{min}$ as a function of α/β (SNR), $T_1 \neq 0$

Figure 9. Simulation and theoretical results of $\sigma_\phi^2|_{min}$ as a function of T_1

Figure 10. Locking time as a function of T_1 in a noise free system, using simulation

Figure 11. $\sigma_\phi^2/\sigma_\phi^{2*}$ as a funct-on of Λ_0/Λ_0^*

Figure 12. $\sigma_\phi^2/\sigma_\phi^{2*}$ as a function of Λ_0/Λ_0^*

Figure 13. Comparing σ_ϕ^2 as a function of T_1 for the optimal system with a system whose phase detector functions are $f_D(\phi) = b_1 \sin \phi$, $f_1(\phi) = a_0 + a_1 \cos \phi$

Figure 14. The sub-optimal system

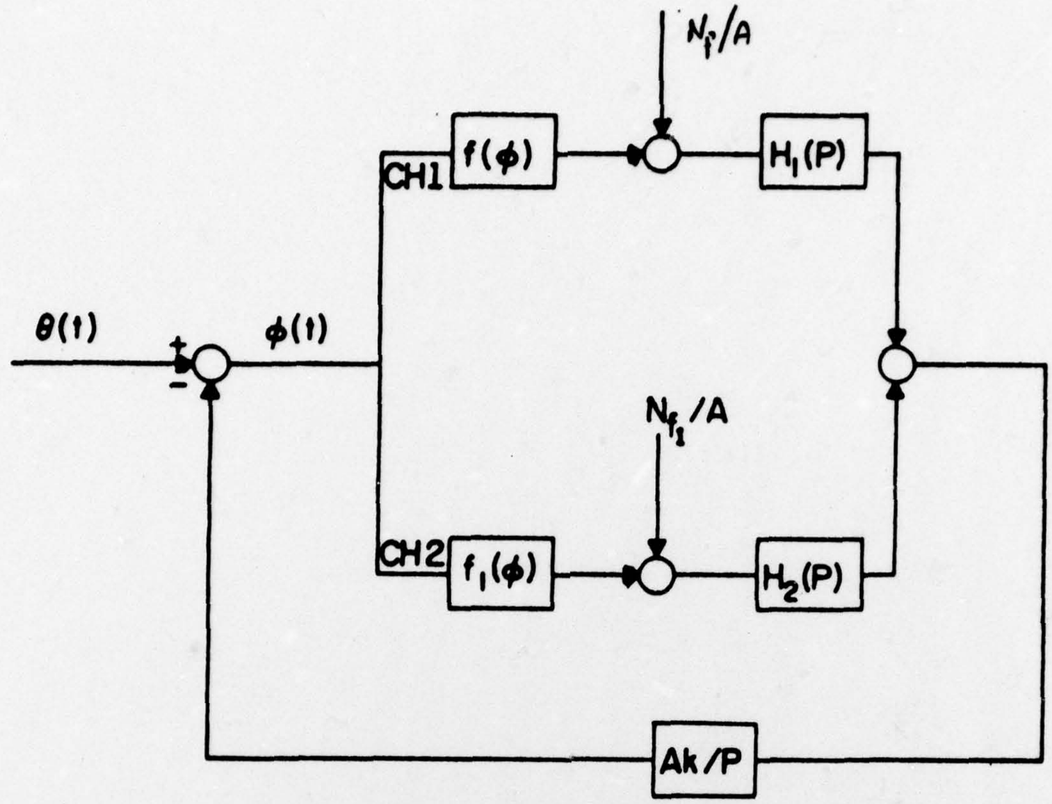


Fig. 1

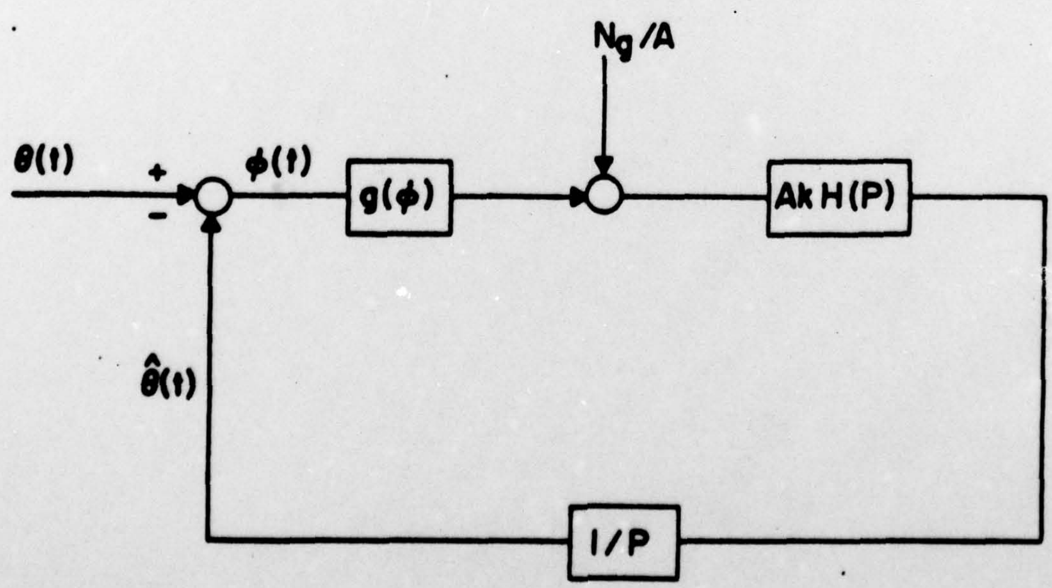


Fig. 2

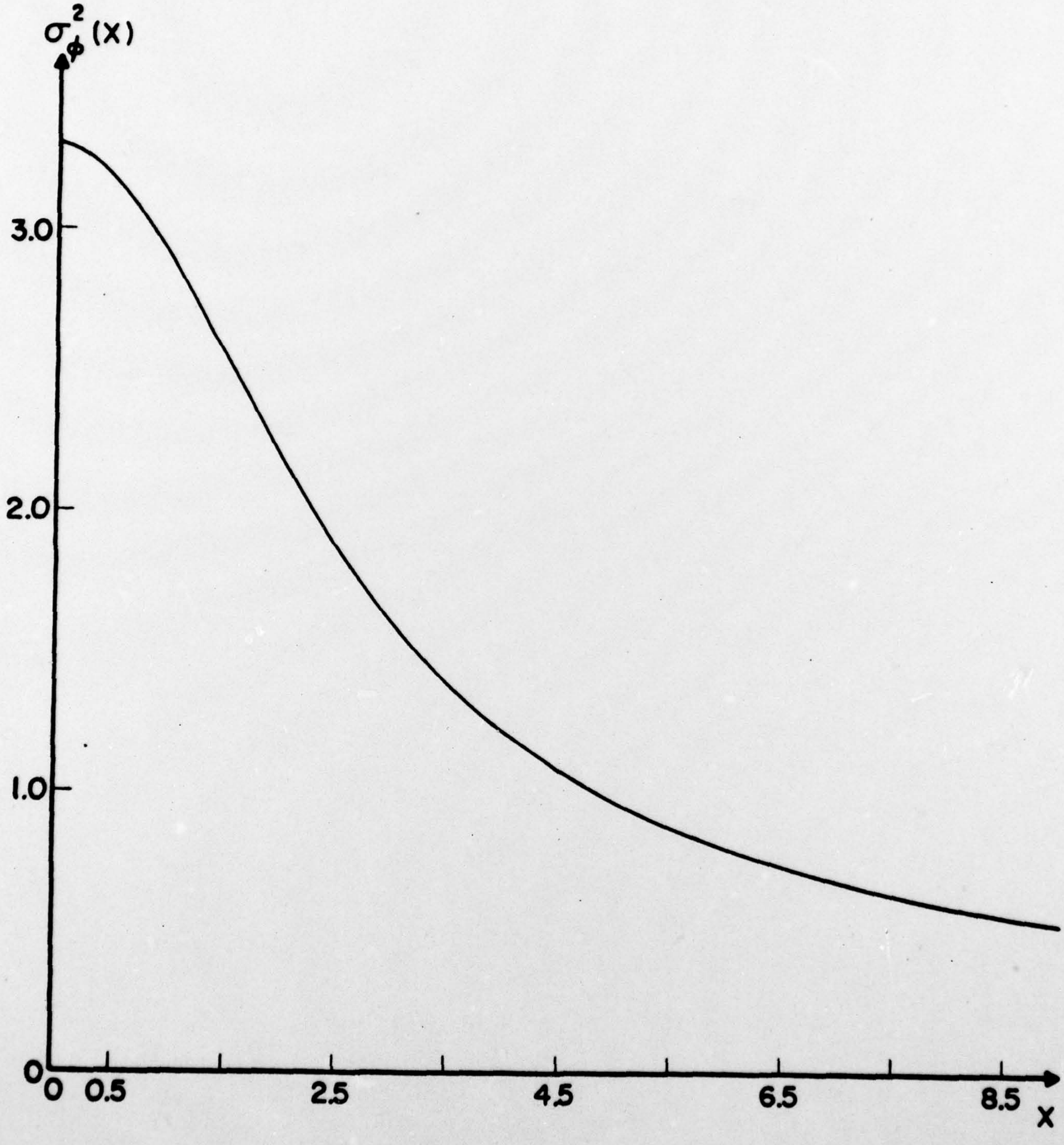
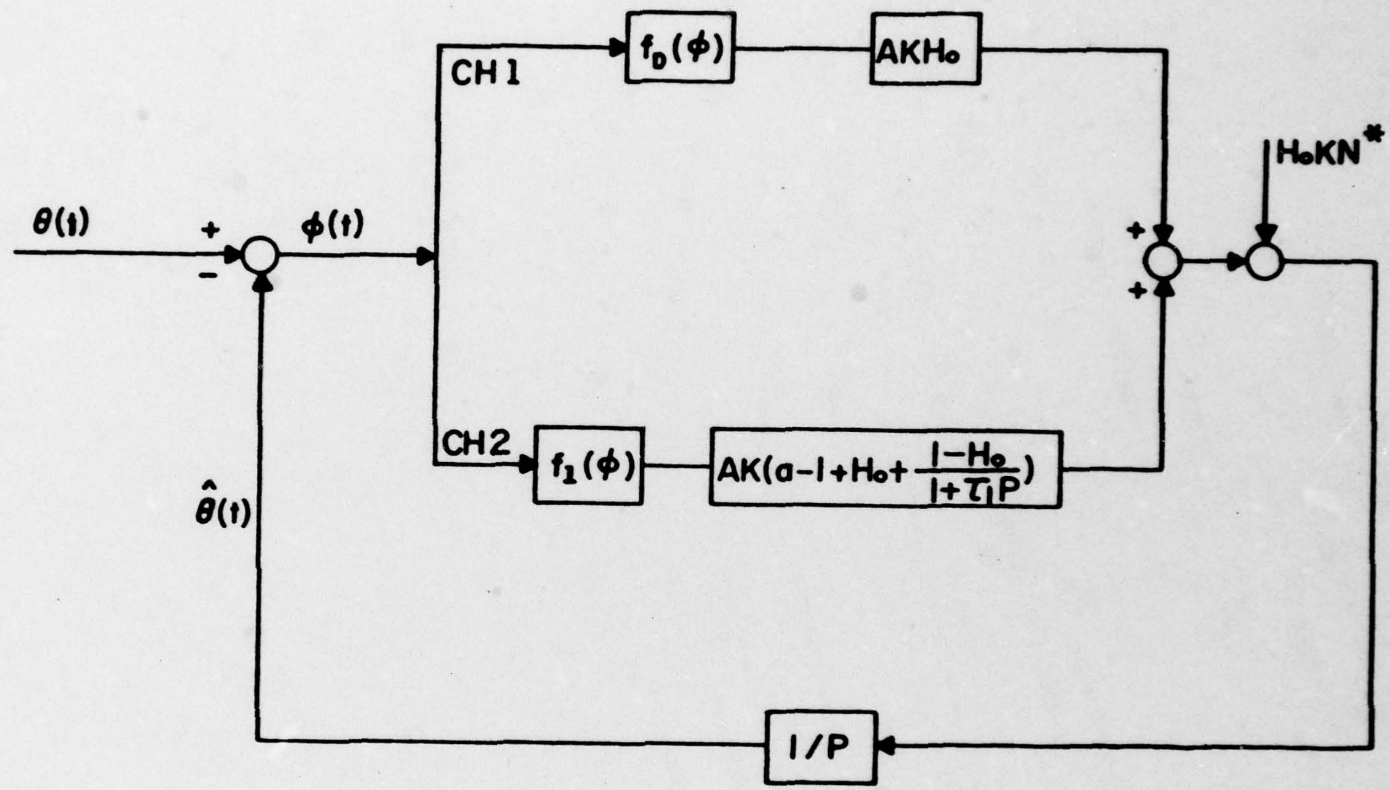
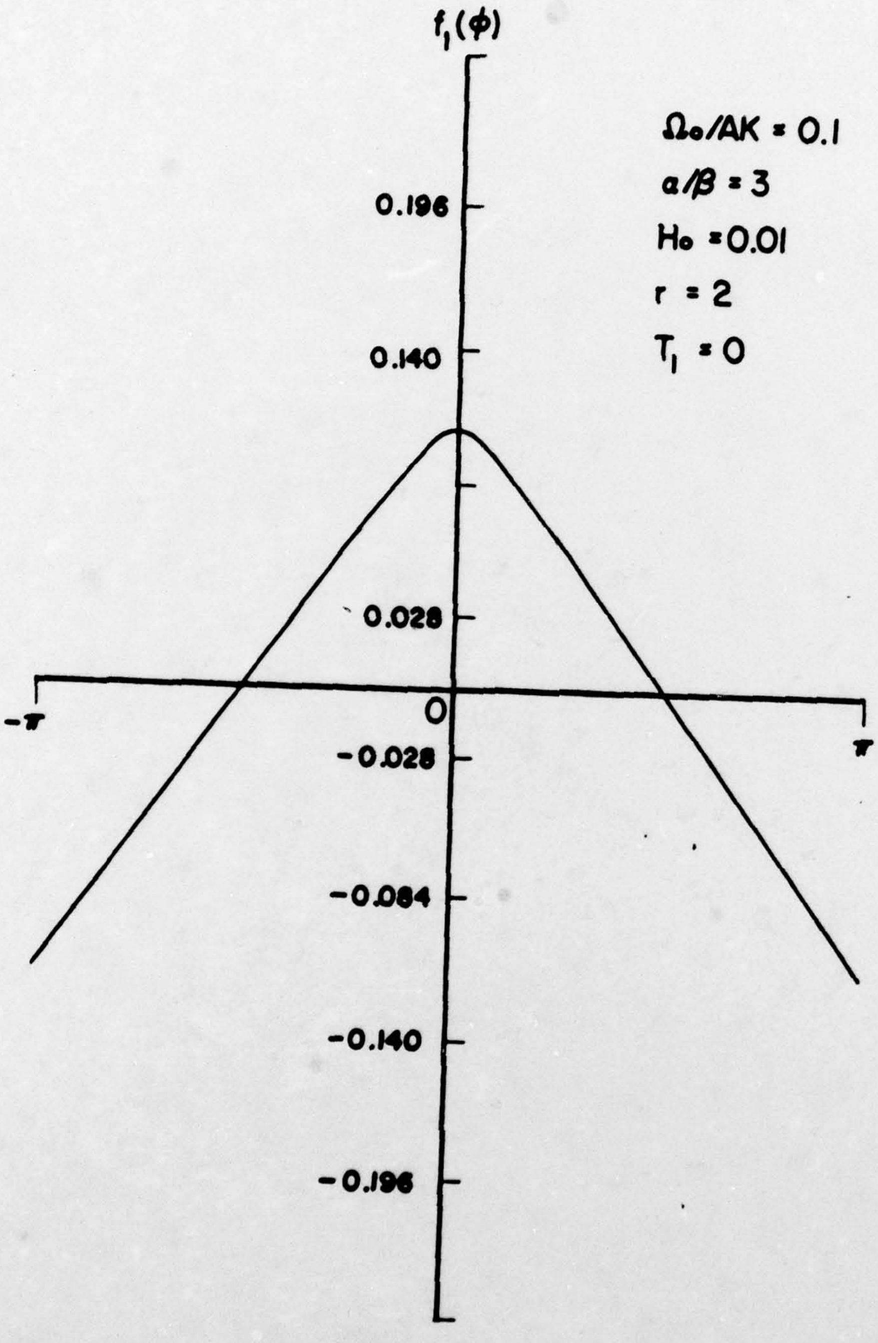


Fig. 3



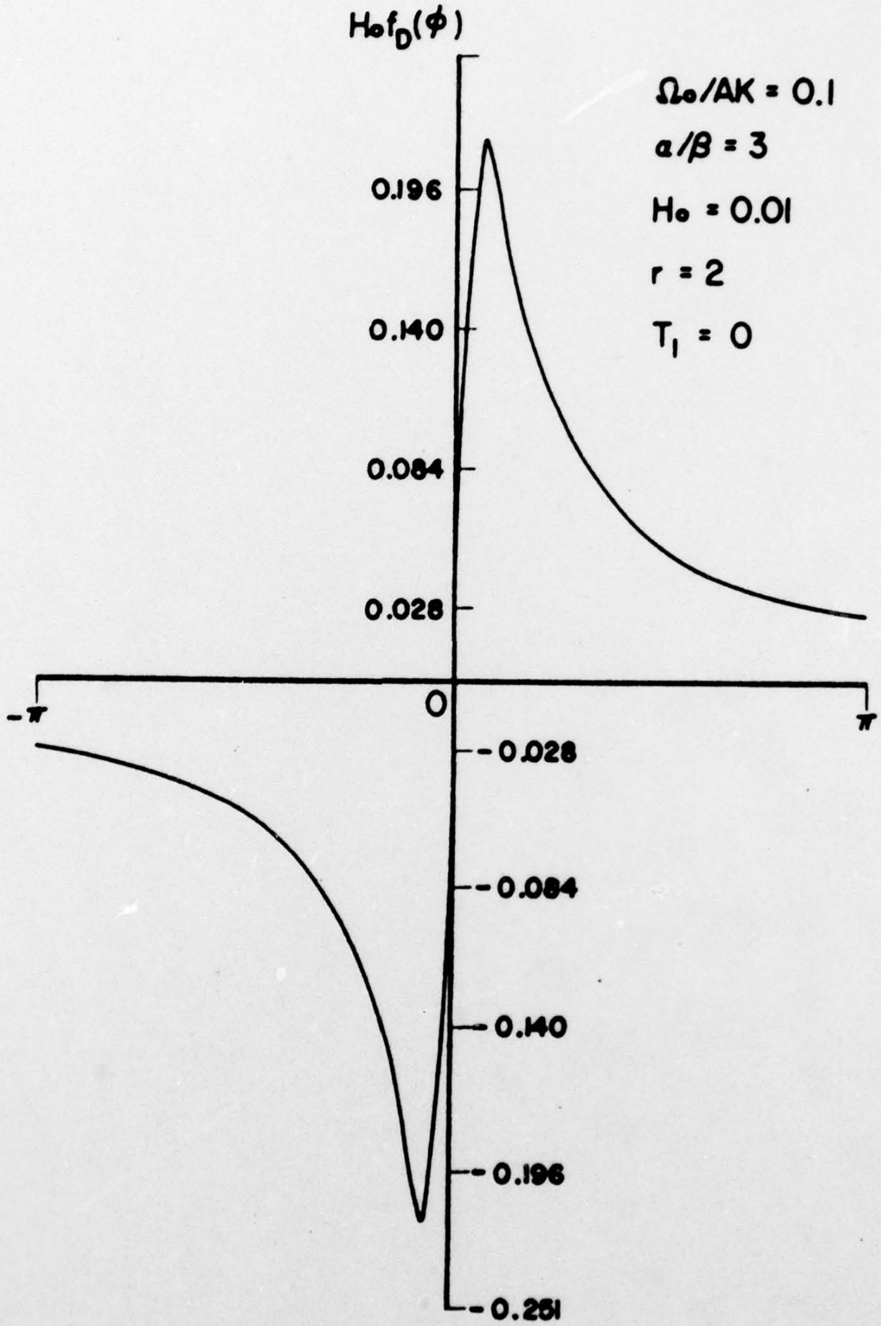
θ(t)
+
−
φ(t)
CH1
CH2
$f_D(\phi)$
AKH_o
$f_1(\phi)$
$AK(a-1+H_o+\frac{1-H_o}{1+\tau_1 P})$
+
+
H_oKN^*
1/P
$\hat{\theta}(t)$

Fig. 4



f1(φ)
Ωo/AK = 0.1
a/β = 3
Ho = 0.01
r = 2
T1 = 0
0.196
0.140
0.028
0
-0.028
-0.084
-0.140
-0.196
-π
π

Fig. 5



$H_o f_D(\phi)$
$\Omega_o/AK = 0.1$
$a/\beta = 3$
$H_o = 0.01$
$r = 2$
$T_1 = 0$
0.196
0.140
0.084
0.028
$-\pi$
0
π
-0.028
-0.084
-0.140
-0.196
-0.251

Fig. 6

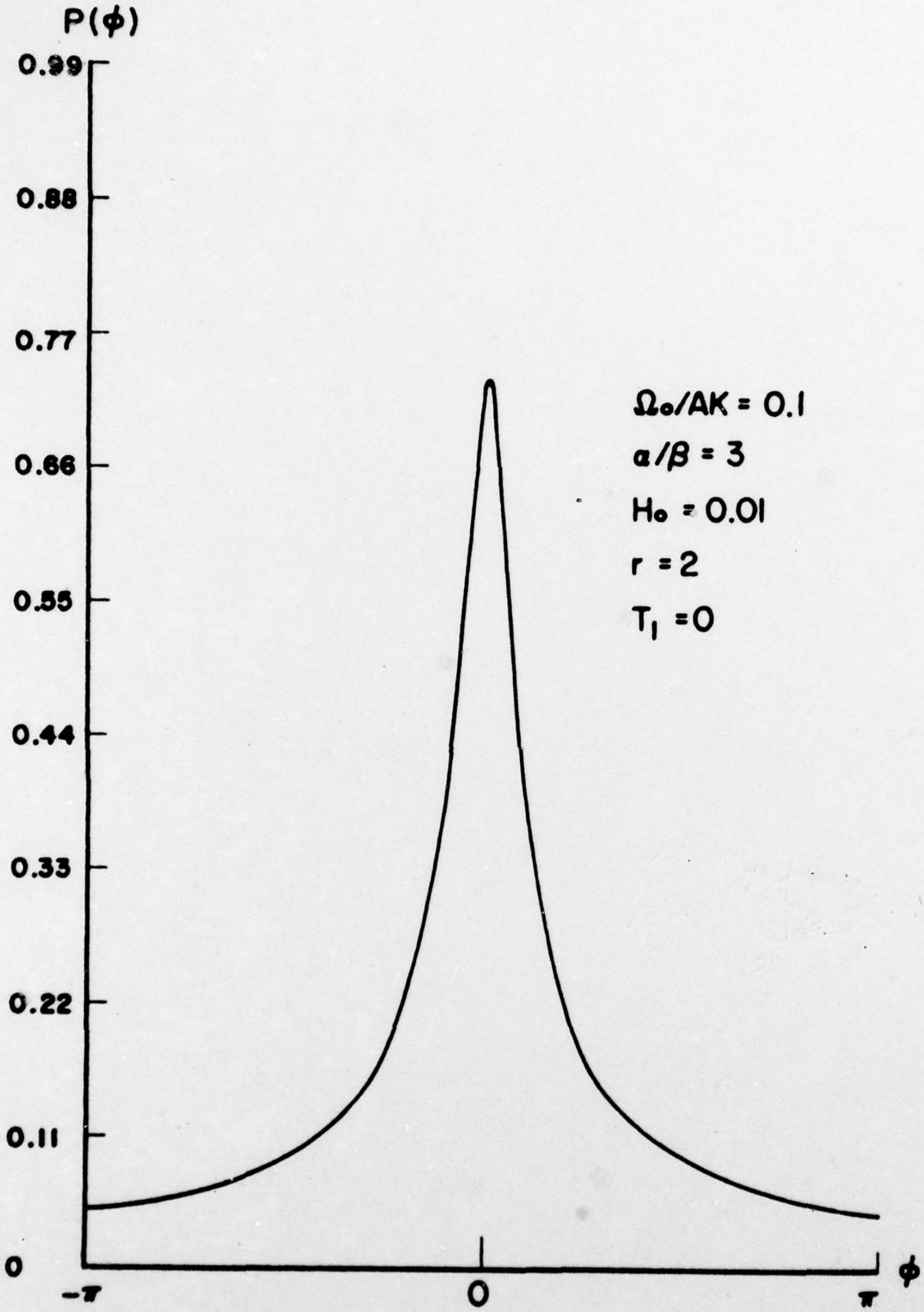

P(ϕ)
0.99
0.88
0.77
0.66
0.55
0.44
0.33
0.22
0.11
0
−π
0
π
ϕ
Ωo/AK = 0.1
α/β = 3
Ho = 0.01
r = 2
T1 = 0

Fig. 7

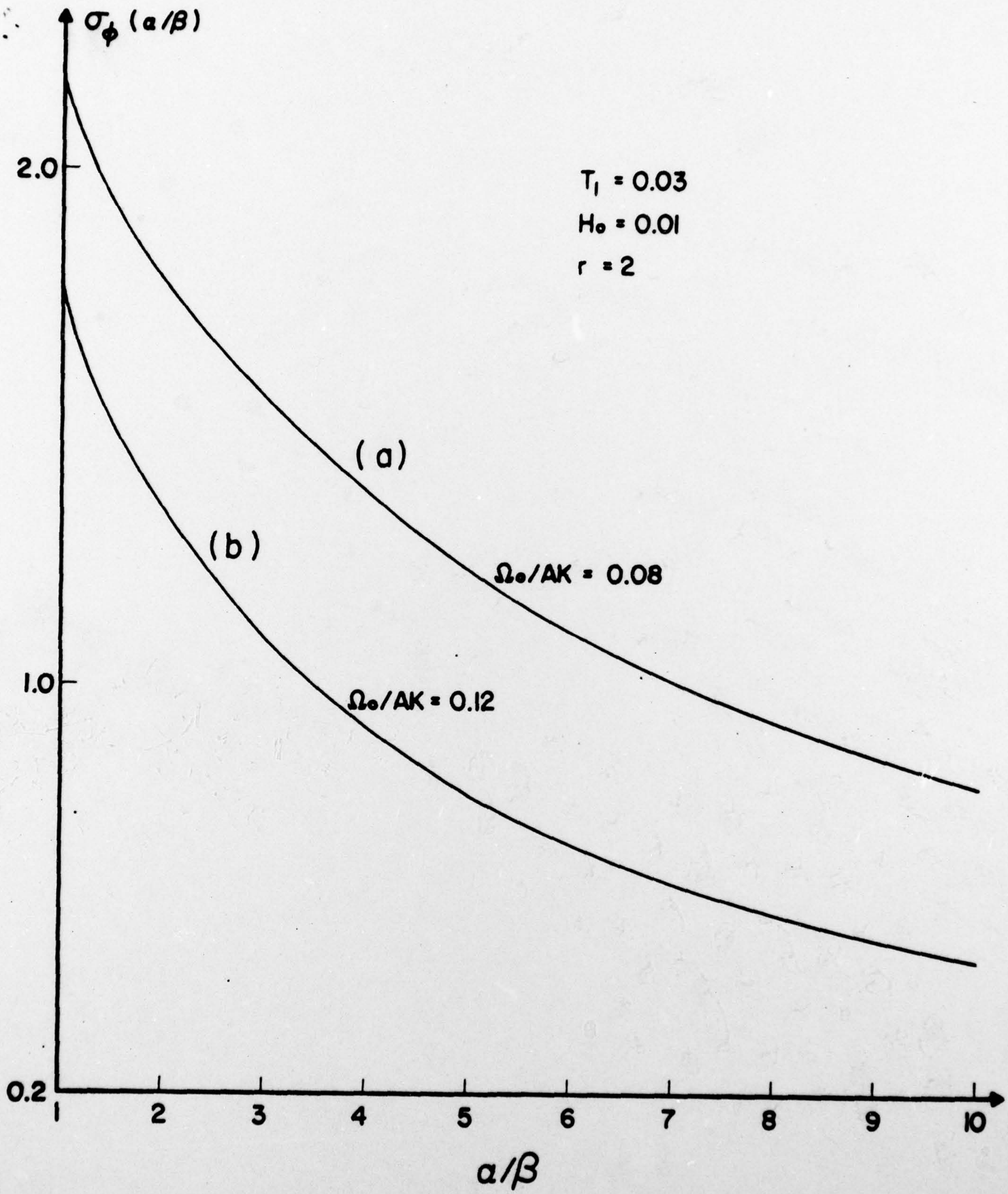
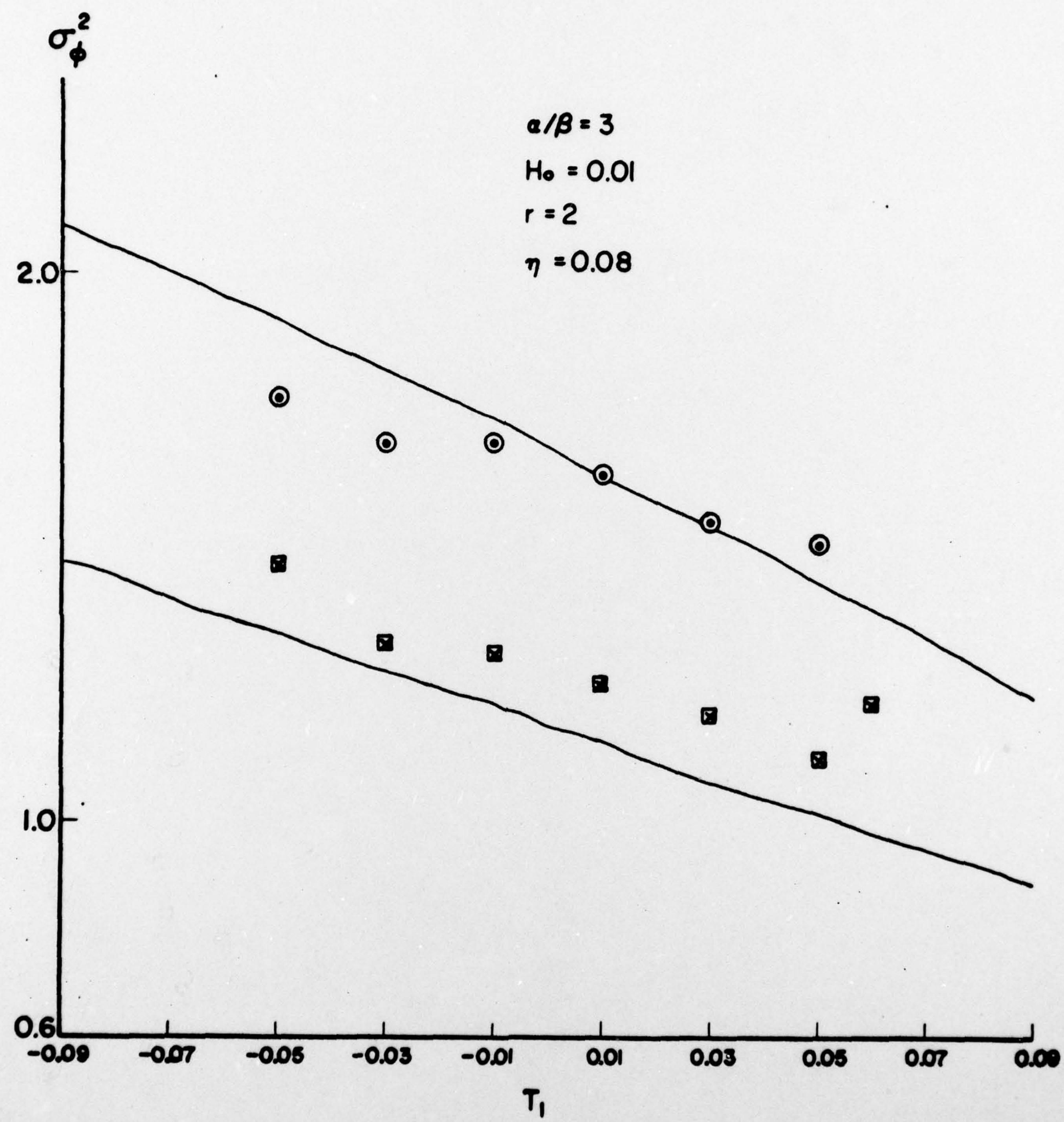
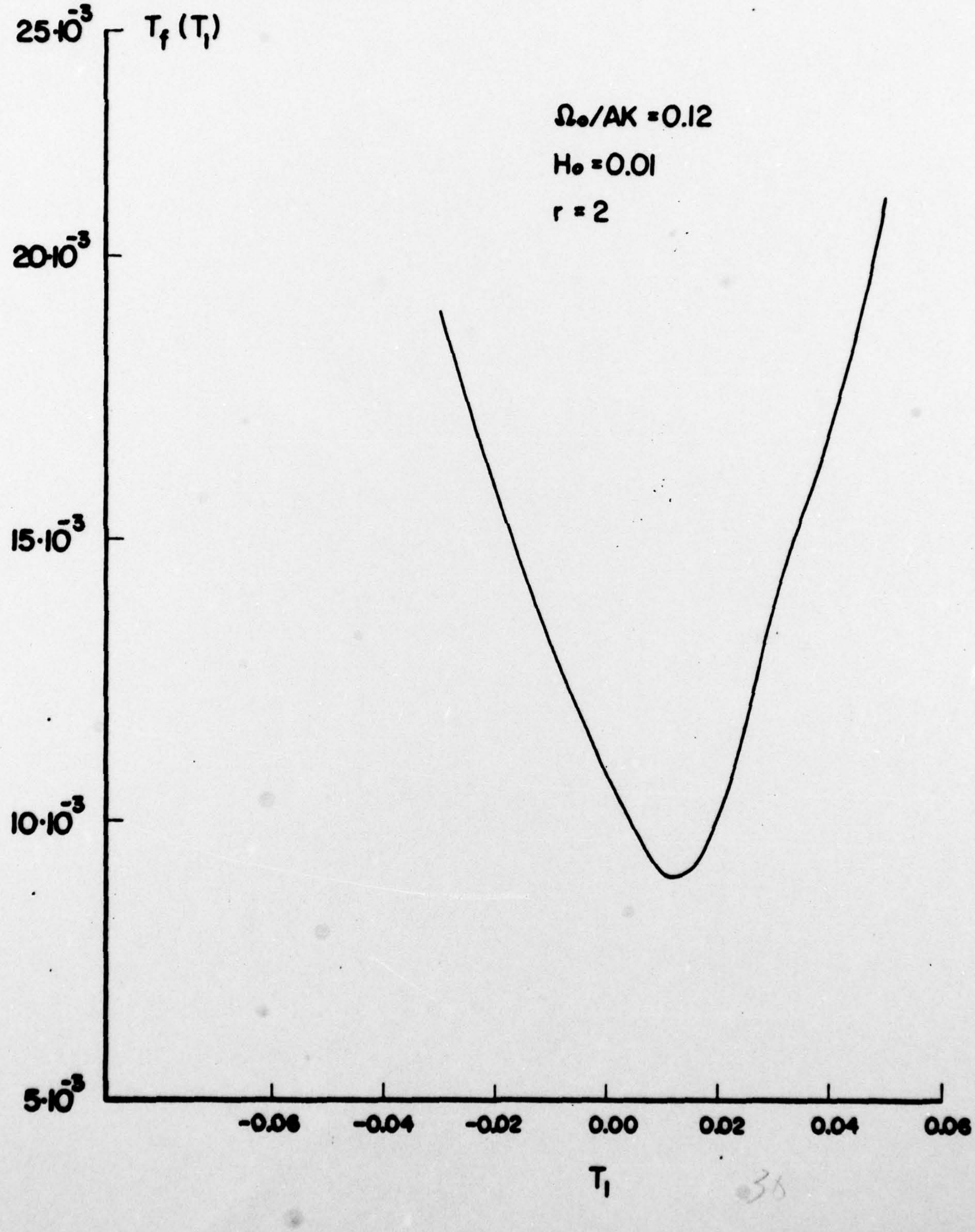

$\sigma_\phi(\alpha/\beta)$
2.0
1.0
0.2
$T_1 = 0.03$
$H_o = 0.01$
$r = 2$
(a)
(b)
$\Omega_o/AK = 0.08$
$\Omega_o/AK = 0.12$
1
2
3
4
5
6
7
8
9
10
α/β

Fig. 8



σ_ϕ^2
$a/\beta = 3$
$H_o = 0.01$
$r = 2$
$\eta = 0.08$
2.0
1.0
0.6
-0.09
-0.07
-0.05
-0.03
-0.01
0.01
0.03
0.05
0.07
0.09
T_1

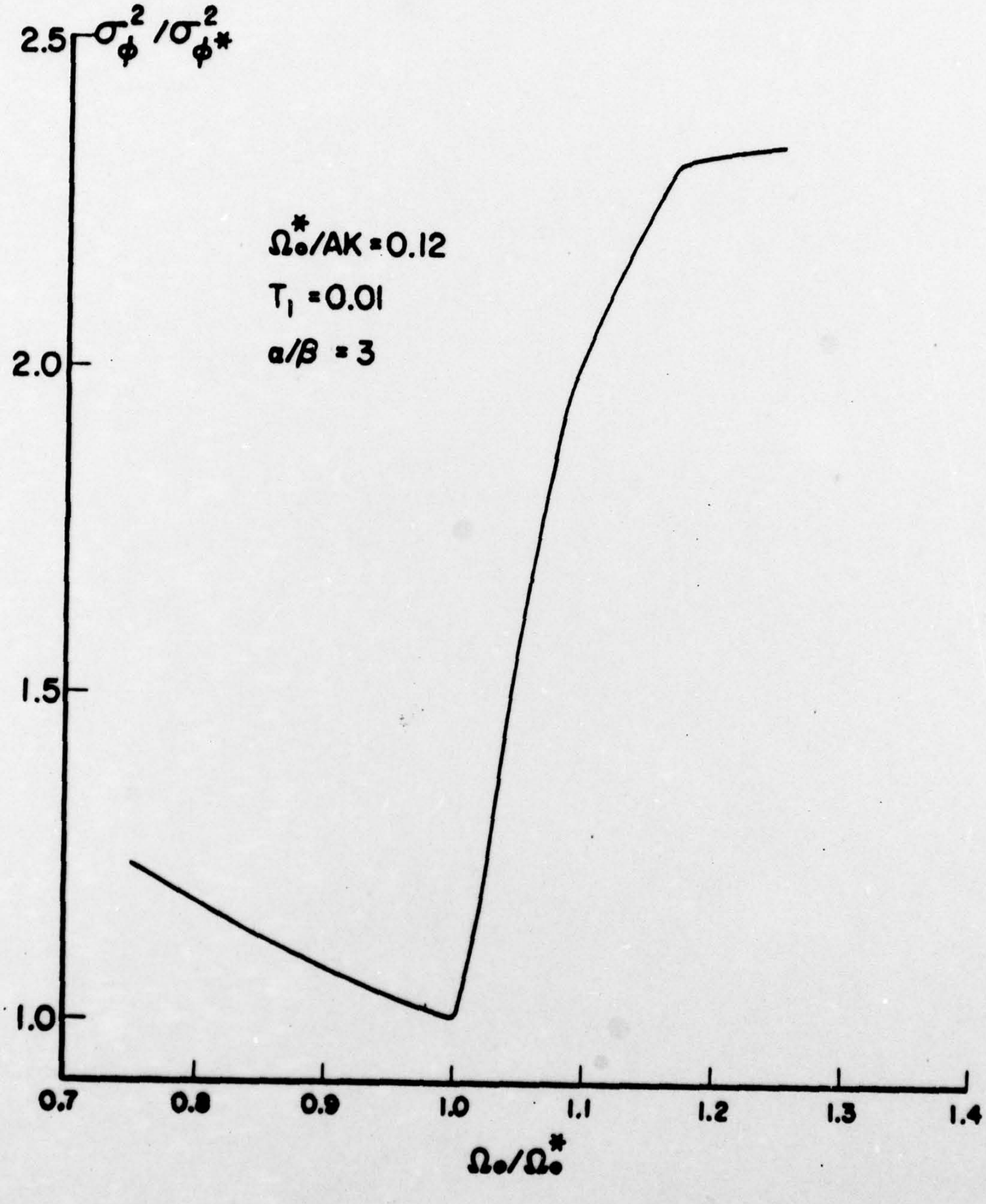
Fig. 9



25·10⁻³
$T_f\,(T_1)$
$\Omega_o/AK = 0.12$
$H_o = 0.01$
$r = 2$
20·10⁻³
15·10⁻³
10·10⁻³
5·10⁻³
-0.06
-0.04
-0.02
0.00
0.02
0.04
0.06
T_1

38

Fig. 10



$\sigma_\phi^2/\sigma_{\phi^*}^2$
2.5
2.0
1.5
1.0
$\Omega_o^*/AK = 0.12$
$T_1 = 0.01$
$\alpha/\beta = 3$
0.7
0.8
0.9
1.0
1.1
1.2
1.3
1.4
Ω_o/Ω_o^*

Fig. 11

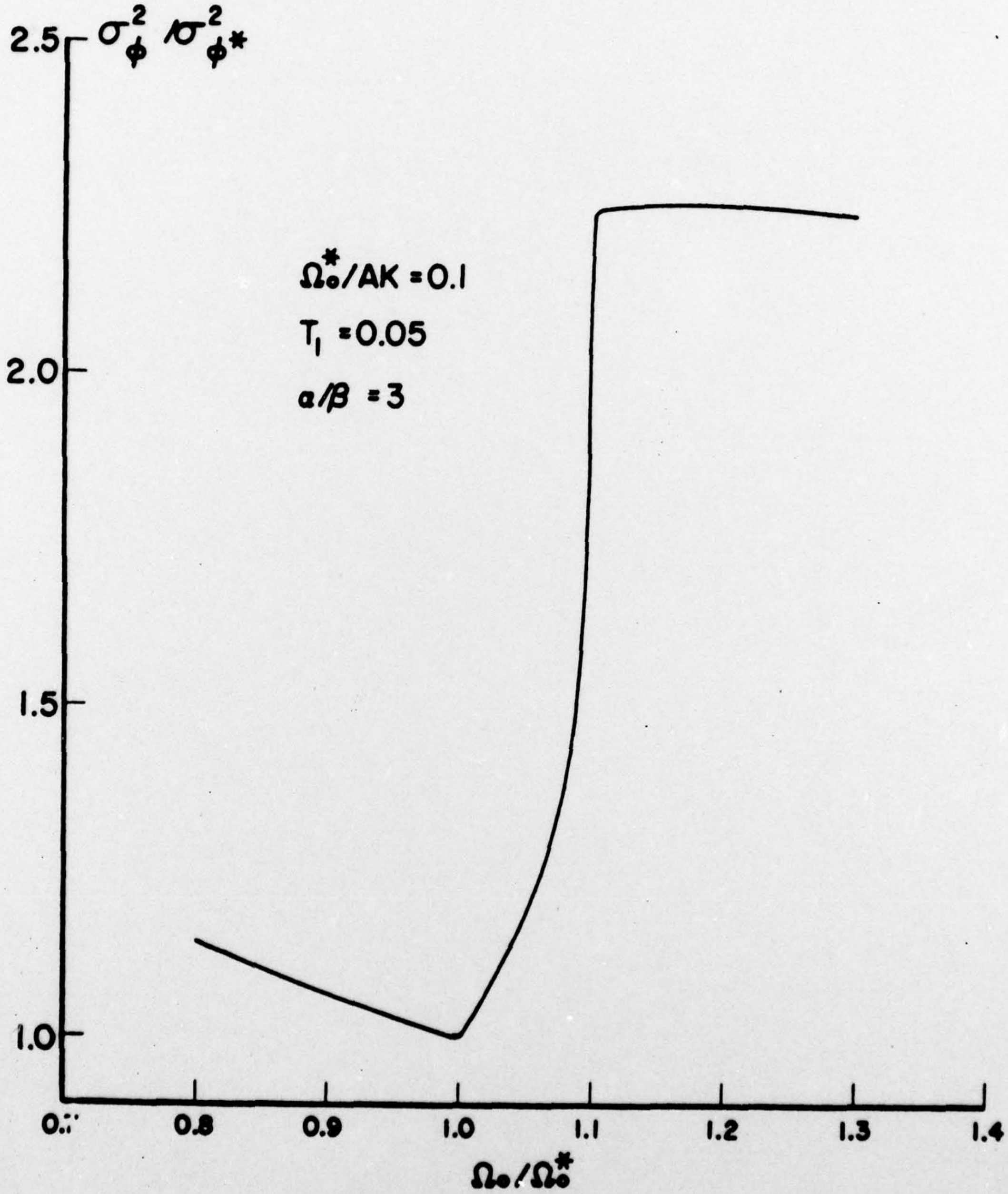
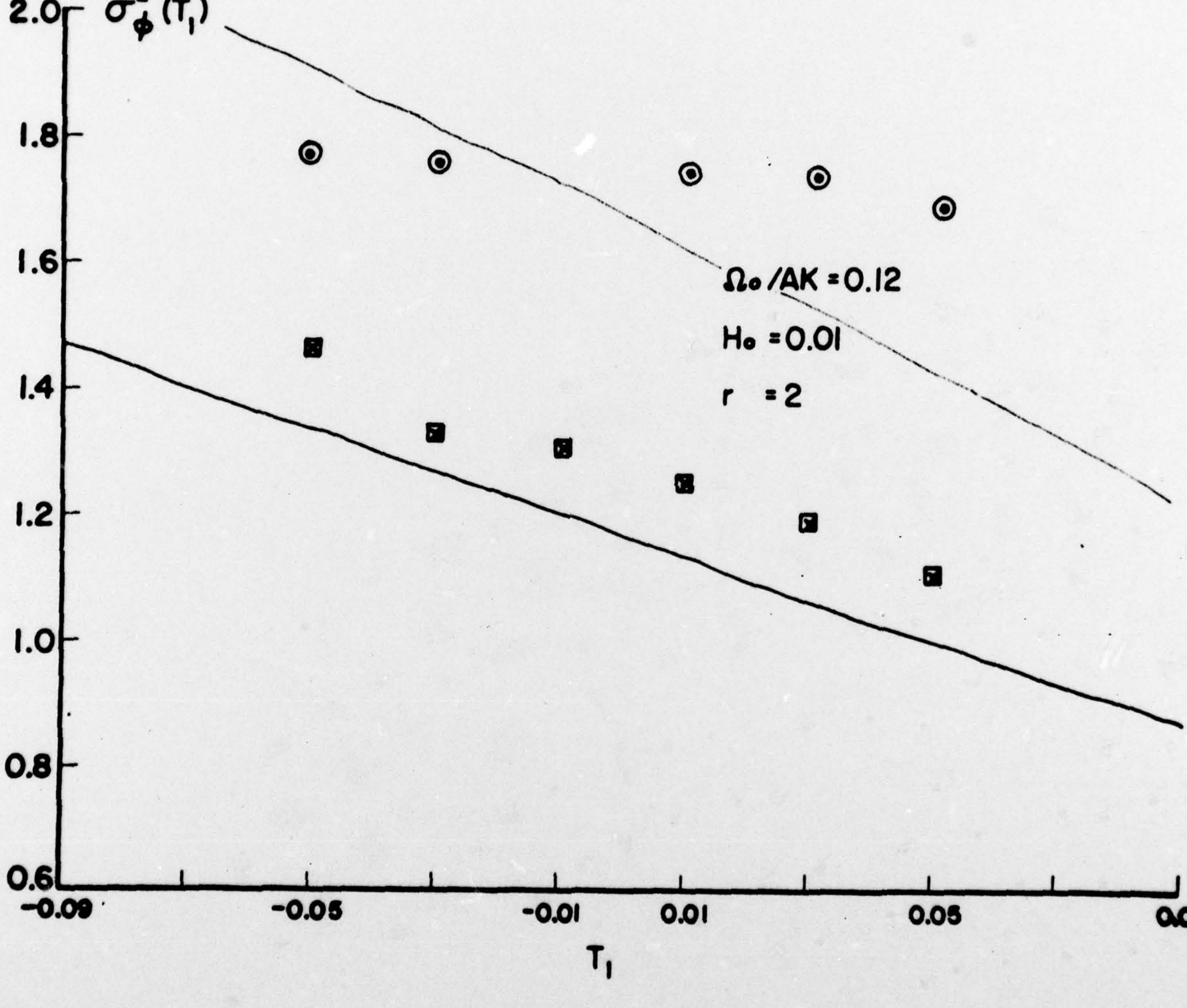

2.5
$\sigma_\phi^2 / \sigma_{\phi^*}^2$
$\Omega_o^*/AK = 0.1$
$T_1 = 0.05$
$\alpha/\beta = 3$
2.0
1.5
1.0
0.8
0.9
1.0
1.1
1.2
1.3
1.4
Ω_o/Ω_o^*

Fig. 12



$\sigma_{\phi}^{2}(T_1)$
2.0
1.8
1.6
1.4
1.2
1.0
0.8
0.6
$\Omega_o/AK = 0.12$
$H_o = 0.01$
$r = 2$
-0.09
-0.05
-0.01
0.01
0.05
0.09
T_1

Fig. 13

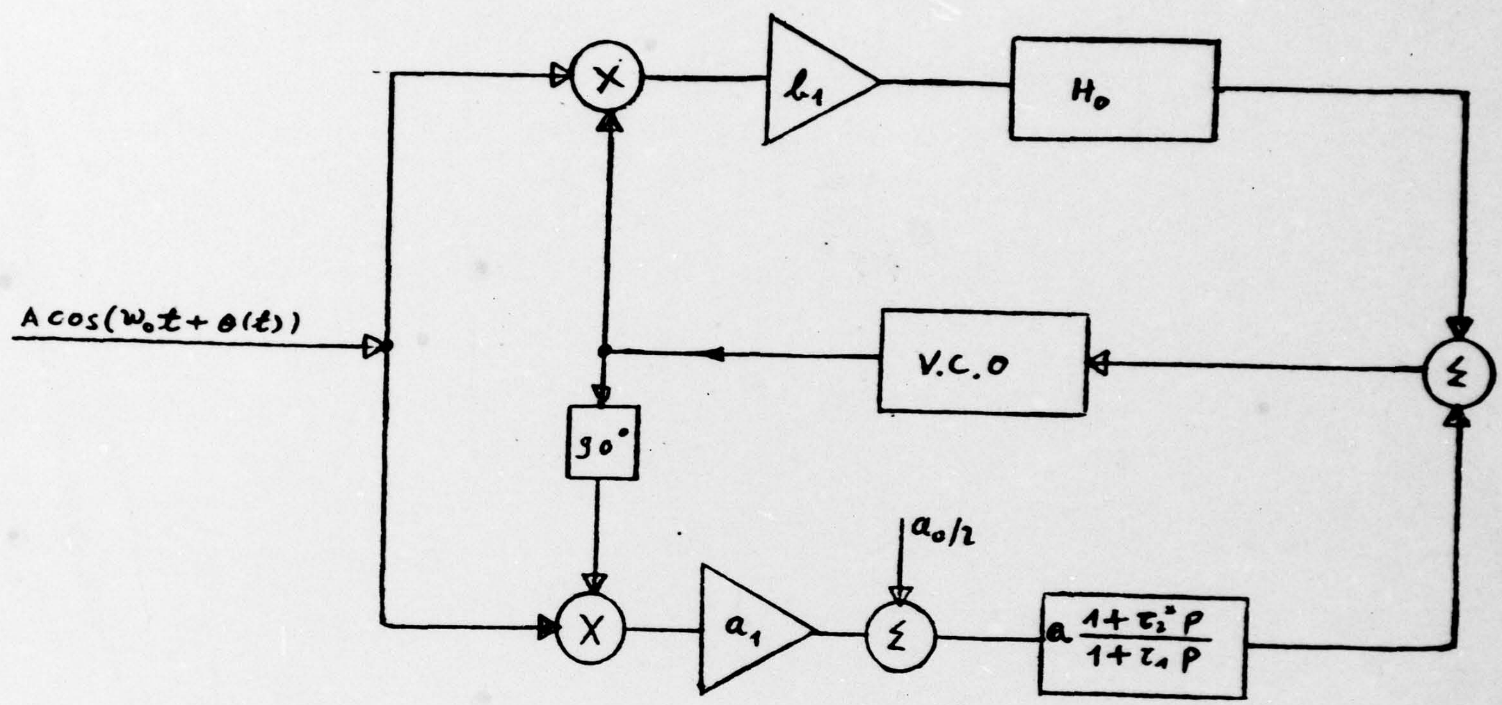

$A\cos(w_0t+\theta(t))$
b_1
H_0
V.C.O
90°
$a_0/2$
a_1
Σ
$a\frac{1+\tau_2^*p}{1+\tau_1 p}$

Fig. 14